ALGEBRA

List 7.

Eigenvalues and eigenvectors. Diagonalization and the Jordan normal form of a matrix

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1. Determine the real eigenvalues and eigenvectors of the following matrices:

$$\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -1 & -2 \\ 0 & 2 & 2 \\ 0 & -1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

2. Find the complex eigenvalues and eigenvectors of the following matrices:

$$\begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & -4 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

3. Find the eigenvalues and eigenvectors of the following linear mappings:

(a) $T : \mathbb{R}^2 \to \mathbb{R}^2$, where $T(x, y) = (x - 2y, x + y)$;

(b) $T : \mathbb{R}^3 \to \mathbb{R}^3$, where $T(x, y, z) = (2z, x, y)$;

(c) $T : \mathbb{C}^3 \to \mathbb{C}^3$, where $T(x, y, z) = (2z, x, y)$;

(d) $T : \mathbb{R}^3 \to \mathbb{R}^3$, where $T(x, y, z) = (2x + y, y - z, z)$.

In each of the above cases, answer whether the mapping has an eigenbasis.

4. For the matrix

$$A = \begin{pmatrix} 2 & -1 & -1 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{pmatrix},$$

(a) determine its eigenvalues and eigenvectors;

(b) check if the matrix is diagonalizable;

(c) give geometric interpretation for the linear transformation given by this matrix

(d) calculate A^{10}, e^A.

5. For the matrix

$$A = \begin{pmatrix} -4 & 6 & -10 \\ -1 & 1 & -2 \\ 1 & -2 & 3 \end{pmatrix},$$

(a) determine its eigenvalues and eigenvectors;

(b) check if the matrix is diagonalizable;

(c) give geometric interpretation for the linear transformation given by this matrix

(d) calculate A^{10}, e^A.

6. For the matrix

$$A = \begin{pmatrix} 3 & 1 & -1 \\ -4 & -2 & 1 \\ 8 & 2 & -3 \end{pmatrix},$$

(a) determine its eigenvalues and eigenvectors;

(b) check if the matrix is diagonalizable;

(c) give geometric interpretation for the linear transformation given by this matrix

(d) calculate A^{10}, e^A.

7. Diagonalize the real matrices

$$\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & -4 \\ 0 & -1 & 0 \\ 2 & -4 & 5 \end{pmatrix}, \quad \begin{pmatrix} 2 & -1 & -1 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} -5 & 0 & -2 \\ 4 & -1 & 2 \\ 4 & 0 & 1 \end{pmatrix},$$

8. For the matrix

$$A = \begin{pmatrix} 5 & 3 & -3 \\ -8 & -6 & 2 \\ 4 & 4 & 1 \end{pmatrix},$$

(a) determine its eigenvalues and eigenvectors;

(b) for each eigenvalue determine its algebraic multiplicity and geometric multiplicity;

(c) explain why the matrix is not diagonizable;

(d) determine its generalized eigenvectors;

(e) write the decomposition of the matrix to the Jordan normal form.

9. For the matrix

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 3 & 1 & -2 \\ 3 & -1 & -1 \end{pmatrix},$$

(a) determine its eigenvalues and eigenvectors;

(b) for each eigenvalue determine its algebraic multiplicity and geometric multiplicity;

(d) determine its generalized eigenvectors;

(e) write the decomposition of the matrix to the Jordan normal form.

10. For the matrix

$$A = \begin{pmatrix} 2 & 2 & -1 \\ 1 & 1 & 0 \\ 2 & -2 & 2 \end{pmatrix},$$

(a) determine its eigenvalues and eigenvectors;

(b) for each eigenvalue determine its algebraic multiplicity and geometric multiplicity;

(d) determine its generalized eigenvectors

(e) write the decomposition of the matrix to the Jordan normal form.